Transformations of Functions

This section will cover how certain transformations affect functions, and their graphs. The transformations we will be looking at all come under the heading of **linear transformations**.

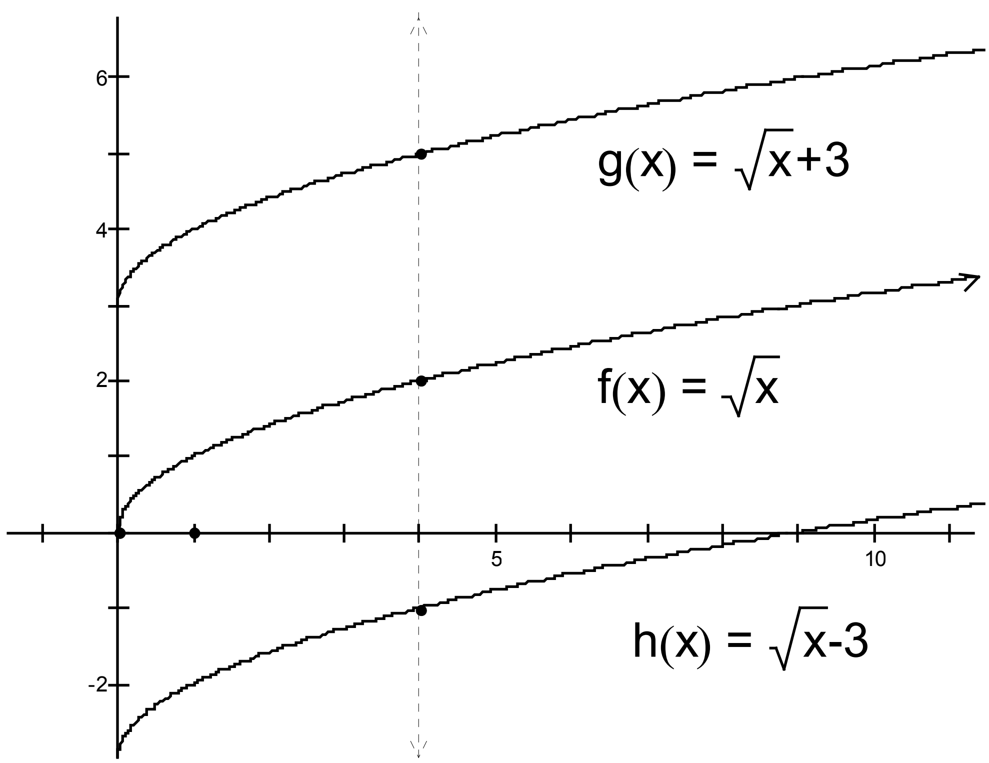
These include vertical and horizontal shifts, reflections, as well as dilations, also called stretching and shrinking.

Vertical Shift

Suppose we have a function $y = f(x)$ and we want to know what happens when we add or subtract a positive constant *C*.

So we could have $y = f(x) + C$ or $y = f(x) - C$. It should be obvious looking at these equations that this transformation will move every point on the graph in the *Y* direction.

Assume for a now that $f(x) = \sqrt{x}$.

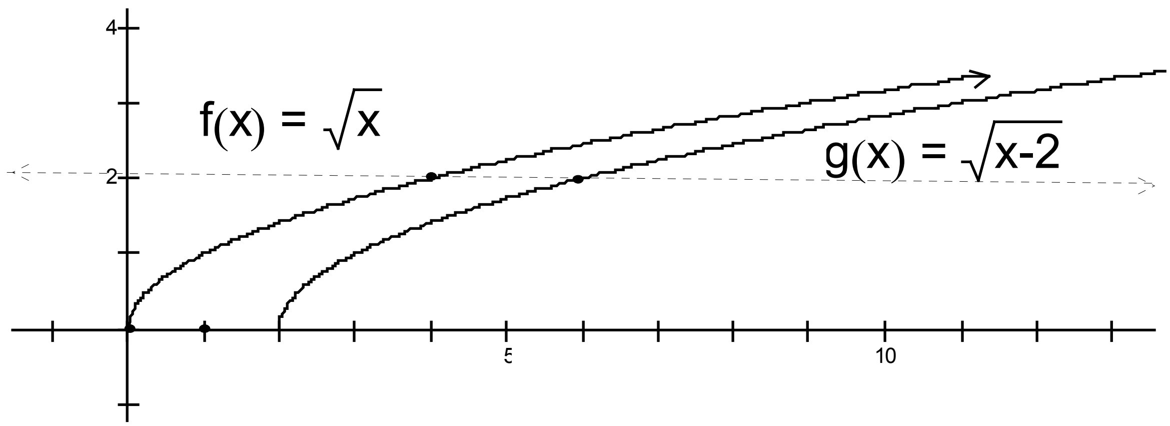


Each vertical line the three functions are each separated by 3.
Adding a constant move the graph upward the amount *C* and subtracting a constant moves the graph down *C*. This type of transformation is the easiest to understand.

Horizontal Shift

The way horizontal shifts work might seem counter-intuitive at first. Let's look at $g(x) = f(x-C) = \sqrt{x-C}$

At first glance one might think that because we are subtracting C from x that it might move the graph to the left. Let's take a look at the graph.



Note that $g(x)$ is a transformation of $f(x)$ and it has moved to the right. If we were to instead add a constant to the x it would move to the left.

Consider just the point $(0, f(0)) = (0,0)$. When we transform this function into $g(x)$, what value do we need to replace x=0 with so that the y coordinate is still $f(0)$?

$f(x) = \sqrt{x}$	$g(x) = f(x-2) = \sqrt{x-2}$
$f(0) = \sqrt{0}$	$g(0+2) = \sqrt{2-2} = \sqrt{0}$

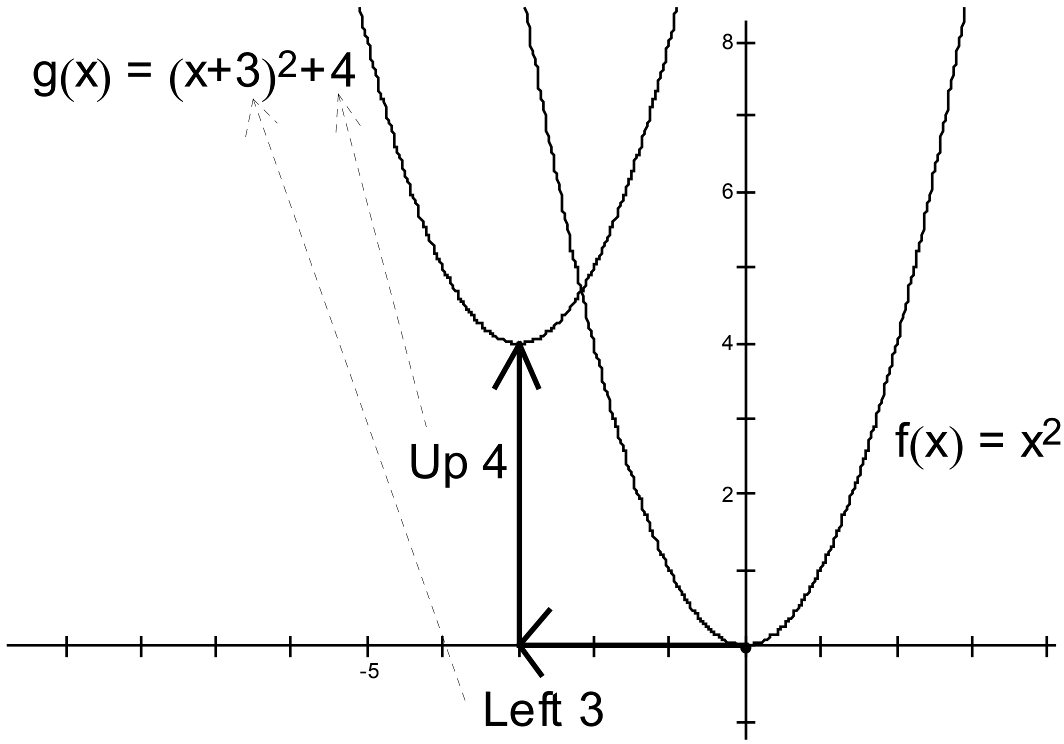
So to get f and g to match in the Y direction, we have to add 2 to every x we give to g.

That is g is moved to the right a distance of 2.

To summarize, the function $g(x) = f(x-v) + h$ is the function $f(x)$ shifted up h and to the right v.

Example:

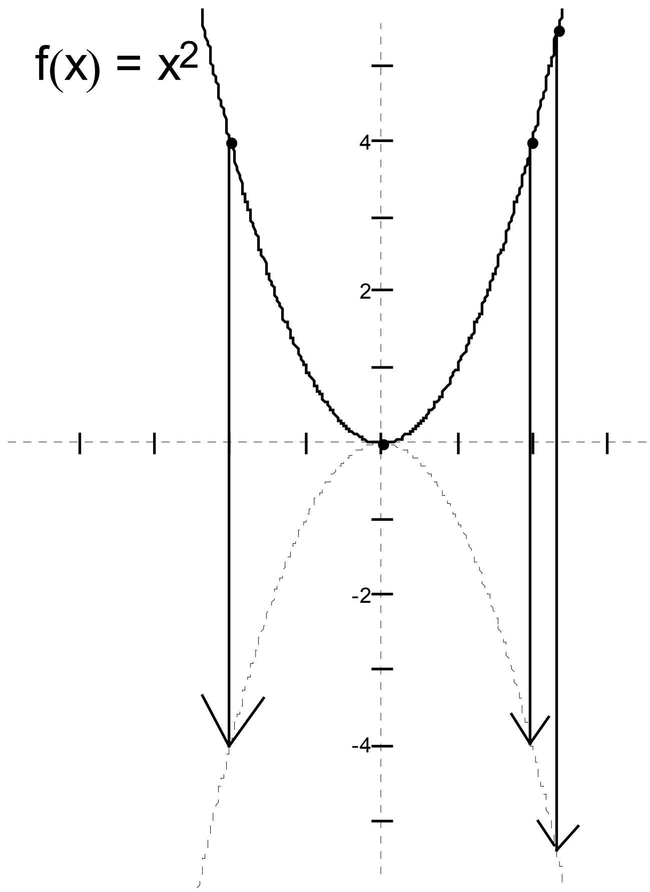
Use the graph of $f(x) = x^2$ to graph the function $g(x) = (x+3)^2 + 4$



Reflections

A reflection is a transformation of each point to a destination on the opposite side of a line, which you can think of as the mirror.

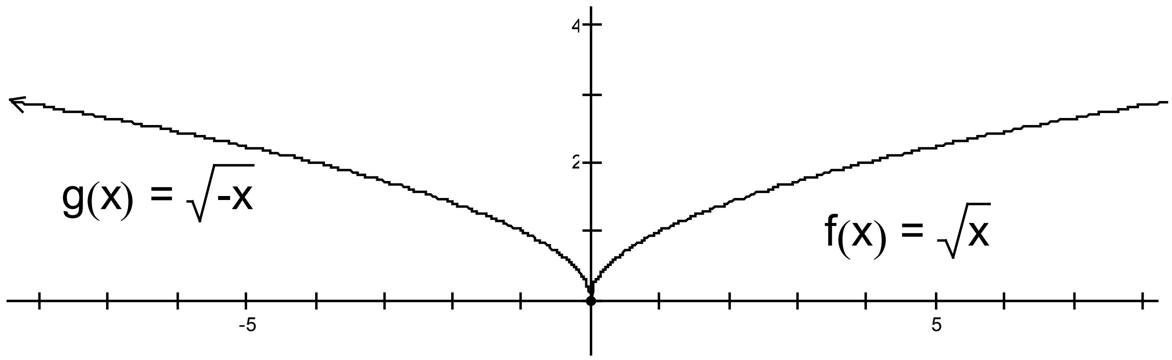
If we take reflect the function $f(x) = x^2$ across the X axis, we get the following.



This is the function $g(x) = -f(x) = -x^2$

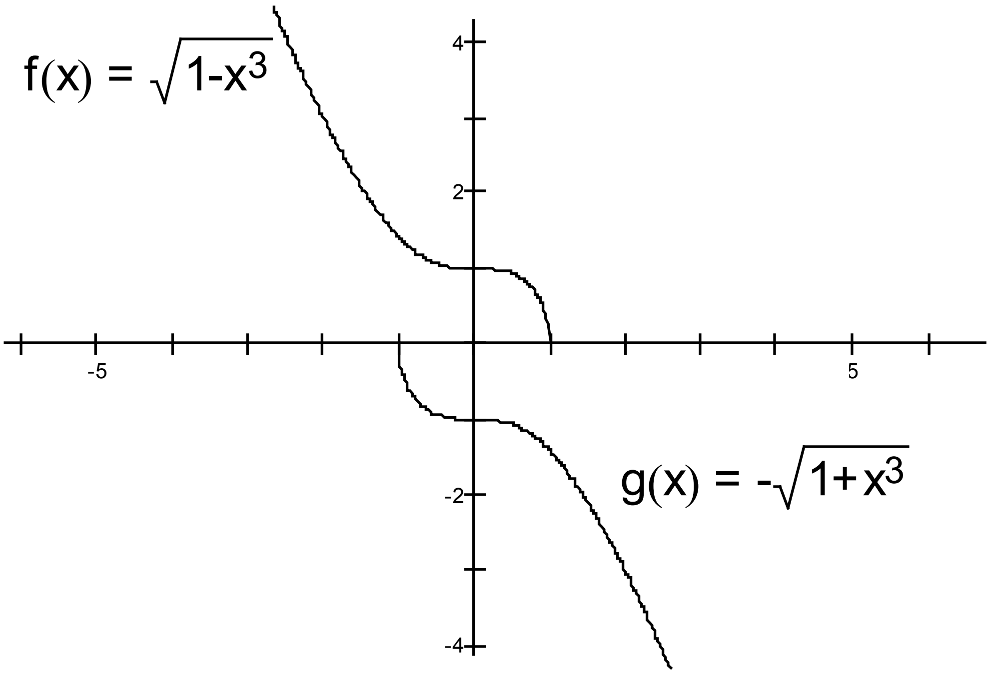
So the function $-f(x)$ is the reflection of the function $f(x)$ across the X-axis.

If we transform the function $f(x)=\sqrt{x}$ to $g(x)=f(-x)=\sqrt{-x}$ the transformation is a reflection across the *Y*-axis.

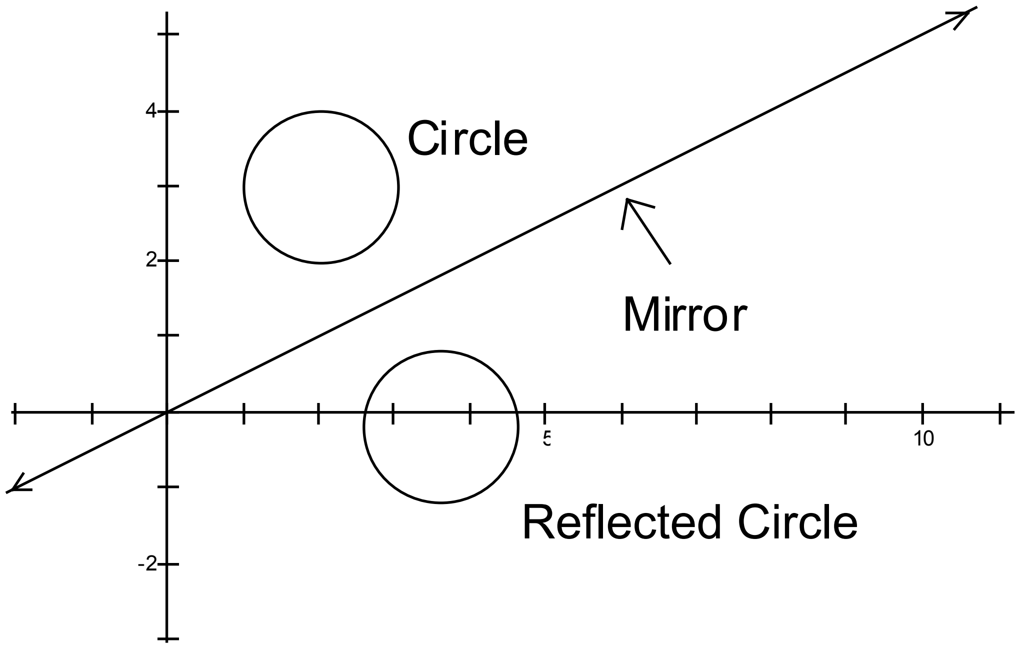


It is also possible to reflect a function $g(x) = -f(-x)$. This will reflect the graph in both the X and Y axis.

Let $f(x) = \sqrt{1-x^3}$ then the function $g(x) = -f(-x) = -\sqrt{1-x^3}$



While we will not be concerned with such reflections, it is possible to reflect a graph across an arbitrary line.

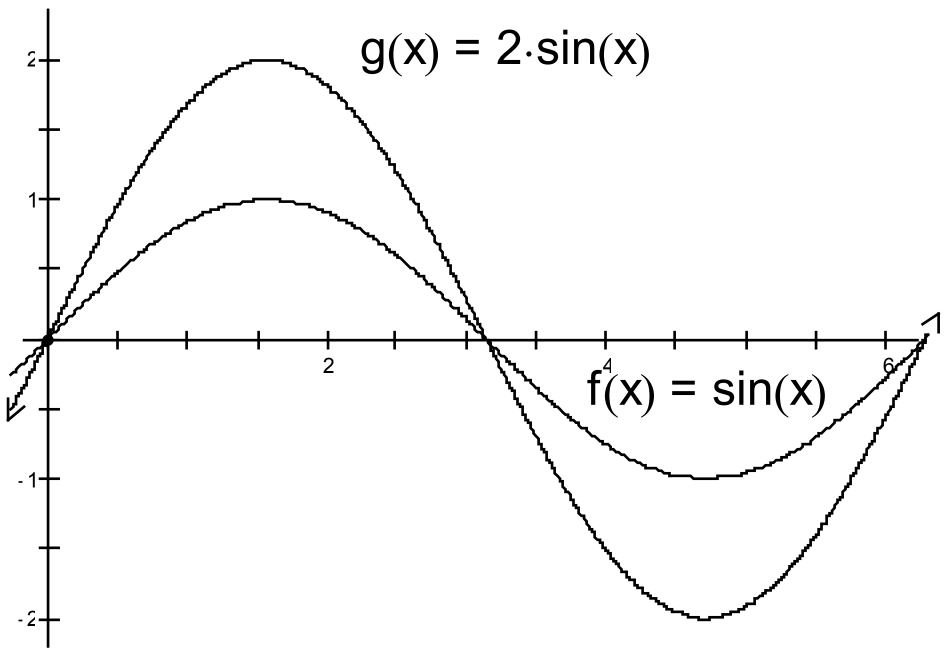


Dilations, Stretching and Shrinking

If one were to transform a function $f(x)$ into $g(x) = Cf(x)$, this would multiply every y coordinate by C, either moving it closer to or away from the Y-axis.

Example:

The function graphed below is called the sine() function. We will see it later in the course. We let $f(x) = \sin(x)$ and transform it to $g(x) = 2f(x) = 2\sin(x)$



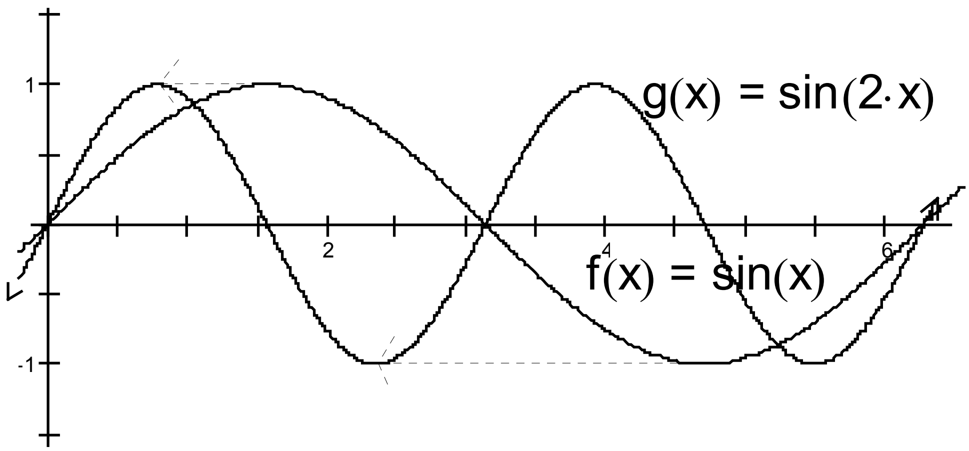
Notice how this stretches the function.

If instead $C<0$ then the graph would shrink in the Y direction.

Like a shift in the *X* direction, dilations in the *X* direction are also counter-intuitive.

Consider the sine function transformed $g\left(x\right)=f\left(2x\right)=\sin\left(2x\right)$.

You might guess that this transformation would stretch the function in the *X* direction, but instead it shrinks it.



Similarly if we transform a function $g\left(x\right)=f\left(Cx\right)$ and 0<C<1 then the function will be stretched in the *Y* direction.

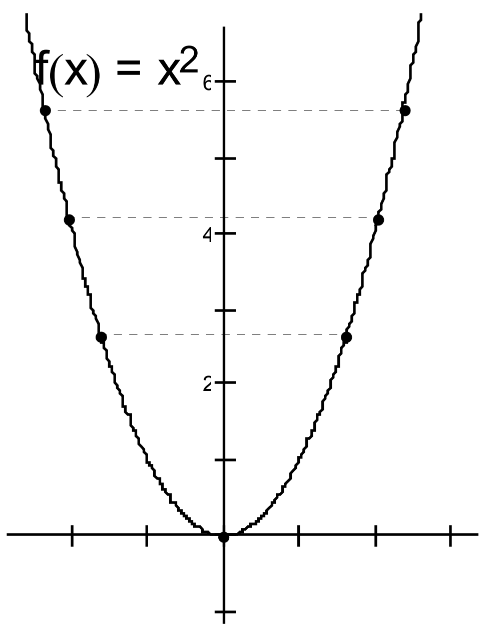
Even and Odd Functions

We are now going to define two important classes of functions, the **even functions** and the **odd functions**. These are functions that display a specific type of symmetry.

One way of understanding an even function, is that it is a function where a reflection across the *Y* axis does not change the function. We've seen this previously when we were looking at symmetry.

So for example, the function $f(x) = x^2$ is an even function because

$$f(-x) = (-x)^2 = (-1)^2 x^2 = x^2 = f(x)$$



Similarly, an odd function is a function that does not change when reflected over both the *X* and *Y* axes does not change. Examples are any functions $f(x) = x^n$ where *n* is odd.

In summary, an even function is one where $f(-x) = f(x)$

An odd function is one where $f(-x) = -f(x)$

Examples:

See whether the following functions are even, odd or neither.

$f(x) = x^5 + x$

$g(x) = 1 - x^4$

$h(x) = 2x - x^2$

Some follow up questions.

1) Can a function be both even and odd?

2) Can a function have symmetry across the X-axis?